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Author: Karol Baron

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ON ORTHOGONALLY ADDITIVE FUNCTIONS
WITH BIG GRAPHS

KAROL BARON

Abstract. Let E be a separable real inner product space of dimension at least 2 and V be a metrizable and separable linear topological space. We show that the set of all orthogonally additive functions mapping E into V and having big graphs is dense in the space of all orthogonally additive functions from E into V with the Tychonoff topology.

Let E be a real inner product space of dimension at least 2 and V a linear topological Hausdorff space. A function f mapping E into V is called *orthogonally additive*, if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in E \text{ with } x \perp y.$$

It is well known, see [6, Corollary 10] (cf. also [4, Theorem 1]), that every orthogonally additive function f defined on E has the form

$$(1) \quad f(x) = a(\|x\|^2) + b(x) \quad \text{for } x \in E,$$

where a and b are additive functions uniquely determined by f .

We continue a study of topological properties of some sets of orthogonally additive functions.

Given a non-empty set S consider the set V^S of all functions from S into V with the usual Tychonoff topology; clearly V^S is a linear topological space.

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In what follows we consider $\text{Hom}_\perp(E, V)$ and $\text{Hom}(S, V)$ for $S \in \{\mathbb{R}, E\}$ with the topology induced by V^E and V^S , respectively, where

$$\text{Hom}_\perp(E, V) = \{f : E \rightarrow V \mid f \text{ is orthogonally additive}\}$$

and

$$\text{Hom}(S, V) = \{f : S \rightarrow V \mid f \text{ is additive}\}$$

for $S \in \{\mathbb{R}, E\}$. According to [2, Corollary 1]:

1. *If $V \neq \{0\}$, then $\text{Hom}(E, V)$ is closed and nowhere dense in $\text{Hom}_\perp(E, V)$.*

This and [1, Theorem] gives:

2. *The set*

$$\{f \in \text{Hom}_\perp(E, V) : f \text{ is injective and } f(E) = V\}$$

is nowhere dense in $\text{Hom}_\perp(E, V)$.

Moreover, see [3, Corollaries 2.3 and 2.4]:

3. *If $\text{card } E \leq \text{card } V$, then the set*

$$\{f \in \text{Hom}_\perp(E, V) : f \text{ is injective and } f(E) \neq V\}$$

is dense in $\text{Hom}_\perp(E, V)$.

4. *If $\text{card } V \leq \text{card } E$, then the set*

$$\{f \in \text{Hom}_\perp(E, V) : f(E) = V \text{ and } f \text{ is not injective}\}$$

is dense in $\text{Hom}_\perp(E, V)$.

The main result of this note concerns density of the set of all orthogonally additive functions with big graphs. Following [5, p. 317] we say that a function f mapping a topological space X into a topological space Y has a *big graph* if $B \cap \text{Graph}(f) \neq \emptyset$ for every Borel subset B of $X \times Y$ such that the projection $\pi_X(B)$ of B onto X has the cardinality $\text{card } X$. We start with the following theorem.

THEOREM 1. *If \mathcal{R} is a family of subsets of $E \times V$ such that*

$$\text{card } \pi_E(R) > \aleph_0 \quad \text{and} \quad \text{card } \pi_E(R) \geq \text{card } \mathcal{R} \quad \text{for } R \in \mathcal{R},$$

then the set

$$\{f \in \text{Hom}_\perp(E, V) : R \cap \text{Graph}(f) \neq \emptyset \text{ for every } R \in \mathcal{R}\}$$

is dense in $\text{Hom}_\perp(E, V)$.

PROOF. Since (see [2, Theorem 1]) the operator which assigns the function f defined by (1) to the variable (a, b) from $\text{Hom}(\mathbb{R}, V) \times \text{Hom}(E, V)$ is a homeomorphism onto $\text{Hom}_\perp(E, V)$, it is enough to prove that the set

$$(2) \quad \bigcap_{R \in \mathcal{R}} \bigcup_{x \in E} \{(a, b) \in \text{Hom}(\mathbb{R}, V) \times \text{Hom}(E, V) : (x, a(\|x\|^2) + b(x)) \in R\}$$

is dense in $\text{Hom}(\mathbb{R}, V) \times \text{Hom}(E, V)$.

We shall show more: if $a \in \text{Hom}(\mathbb{R}, V)$ and $\mathcal{U} \subset \text{Hom}(E, V)$ is open and non-empty, then $\{a\} \times \mathcal{U}$ intersects set (2). To this end we may assume that

$$\mathcal{U} = \{b \in \text{Hom}(E, V) : b(z_n) \in U_n \text{ for } n \in \{1, \dots, N\}\}$$

with some open subsets U_1, \dots, U_N of V , $z_1, \dots, z_N \in E$ and $N \in \mathbb{N}$.

Fix $b_0 \in \mathcal{U}$. To prove that $\{a\} \times \mathcal{U}$ meets set (2) it is enough to show that there is a $b \in \text{Hom}(E, V)$ such that (a, b) is in the set (2) and

$$(3) \quad b(z_n) = b_0(z_n) \quad \text{for } n \in \{1, \dots, N\}.$$

Let H be a base of the vector space E over the field \mathbb{Q} of all rationals and let H_0 be a finite subset of H such that $z_1, \dots, z_N \in \text{Lin}_{\mathbb{Q}} H_0$. Put

$$\gamma = \text{card } \mathcal{R}$$

and let $(R_\alpha)_{\alpha < \gamma}$ be a transfinite sequence of all elements of \mathcal{R} . (We treat γ as an ordinal.) By transfinite induction we define an injective transfinite sequence $(x_\alpha)_{\alpha < \gamma}$ of vectors of E and a transfinite sequence $(v_\alpha)_{\alpha < \gamma}$ of vectors of V such that for every $\alpha < \gamma$ we have

$$(4) \quad H_0 \cup \{x_\beta : \beta \leq \alpha\} \text{ is linearly independent over } \mathbb{Q},$$

$$(5) \quad H_0 \cap \{x_\beta : \beta \leq \alpha\} = \emptyset,$$

and

$$(6) \quad (x_\alpha, v_\alpha) \in R_\alpha.$$

We proceed as follows. If $\delta < \gamma$, $(x_\alpha)_{\alpha < \delta}$ is an injective transfinite sequence of vectors of E and $(v_\alpha)_{\alpha < \delta}$ is a transfinite sequence of vectors of V such that (4)–(6) hold for every $\alpha < \delta$, then

$$\begin{aligned} \text{card } \text{Lin}_{\mathbb{Q}}(H_0 \cup \{x_\alpha : \alpha < \delta\}) &\leq \aleph_0 \cdot \max\{\aleph_0, \text{card } \delta\} \\ &= \max\{\aleph_0, \text{card } \delta\} < \text{card } \pi_E(R_\delta), \end{aligned}$$

and so the set

$$\pi_E(R_\delta) \setminus \text{Lin}_{\mathbb{Q}}(H_0 \cup \{x_\alpha : \alpha < \delta\})$$

is non-empty; choosing a point x_δ from this set we see that $H_0 \cup \{x_\alpha : \alpha \leq \delta\}$ consists of vectors linearly independent over \mathbb{Q} and $(x_\delta, v_\delta) \in R_\delta$ for some $v_\delta \in V$.

Let $b: E \rightarrow V$ be an additive function such that

$$b|_{H_0} = b_0|_{H_0} \quad \text{and} \quad b(x_\alpha) = v_\alpha - a(\|x_\alpha\|^2) \quad \text{for } \alpha < \gamma.$$

Then (3) holds and

$$(x_\alpha, a(\|x_\alpha\|^2) + b(x_\alpha)) = (x_\alpha, v_\alpha) \in R_\alpha$$

for $\alpha < \gamma$ and so (a, b) is in the set (2). \square

COROLLARY 1. *If E is separable and V is metrizable and separable, then the set*

$$(7) \quad \{f \in \text{Hom}_\perp(E, V) : f \text{ has a big graph}\}$$

is dense in $\text{Hom}_\perp(E, V)$.

PROOF. If \mathcal{R} denotes the family

$$\{B \subset E \times V : B \text{ is Borel and } \text{card } \pi_E(B) = \text{card } E\},$$

then (see, e.g., [5, Theorem 2.3.4])

$$\text{card } \mathcal{R} = \mathfrak{c} = \text{card } E = \text{card } \pi_E(B)$$

for every $B \in \mathcal{R}$. \square

Functions with big graphs have a lot of interesting properties. In particular the following (see [5, Theorems 2.5.6 and 2.8.3] and the proofs of Theorems 12.4.4 and 12.4.5 in [5]):

5. *Assume X and Y are Polish spaces.*

- (i) *If X has no isolated points, then the graph of any function with big graph mapping X into Y is dense in $X \times Y$.*
- (ii) *If X and Y have no isolated points and $f: X \rightarrow Y$ has a big graph, then neither $\text{Graph}(f)$, nor $(X \times Y) \setminus \text{Graph}(f)$ contains a second category set with the Baire property.*

- (iii) *If X and Y are connected, then the graph of any function with big graph mapping X into Y is connected.*

Hence and from Corollary 1 we obtain the following corollary.

COROLLARY 2. *If E is separable and Hilbert and V is Polish, $V \neq \{0\}$, then the set*

$$\{f \in \text{Hom}_\perp(E, V) : \text{Graph}(f) \text{ is dense and connected,} \\ \text{and neither } \text{Graph}(f), \text{ nor } (E \times V) \setminus \text{Graph}(f) \\ \text{contains a second category set with the Baire property}\}$$

is dense in $\text{Hom}_\perp(E, V)$.

At the end note also that:

6. *If $V \neq \{0\}$, then the complement of the set (7) is dense in $\text{Hom}_\perp(E, V)$.*

In fact, an obvious modification of the proof of Theorem 1 shows that:

7. *The set*

$$\{f \in \text{Hom}_\perp(E, V) : f(E) \text{ is countable}\}$$

is dense in $\text{Hom}_\perp(E, V)$.

The reader interested in further problems connected with orthogonal additivity is referred to the survey paper [7] by Justyna Sikorska.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF SILESIA
BANKOWA 14
40-007 KATOWICE
POLAND
e-mail: baron@us.edu.pl